Johns Hopkins Math Tournament 2021

Individual Round: Probability and Combinatorics

April 3rd, 2021

Instructions

- **Remember you must be proctored while taking the exam.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No outside help is allowed. This includes people, the internet, translators, books, notes, calculators, or any other computational aid. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor.
- Good luck!

1. Dan has a fair 6-sided die with faces labeled $1, 2, 3, 4, +$, and $-$. In order to complete the equation

$$\_\_\_\_\ \_\_\_\_\ \_\_\_\_ = \_\_\_\_,$$

Dan repeatedly rolls his die and fills in a blank with the character he obtained, starting with the leftmost blank and progressing rightward. The probability that, when all blanks are filled, Dan forms a true equation, is $\frac{p}{q}$, where p an q are relatively prime integers. Find $p + q$.

2. Call a positive integer *almost square* if it is not a perfect square, but all of its digits are perfect squares. For example, both 149 and 904 are almost square, but 144 and 936 are not. Find the number of positive integers less than 1000 that are not almost square.

3. Let $P = (x, y)$ be the coordinates of a point chosen uniformly at random within the unit square (i.e. the square with vertices at $(0, 0), (0, 1), (1, 0)$, and $(1, 1)$). The probability that $|x - \frac{1}{2}| + |y - \frac{1}{2}| < \frac{1}{2}$ is given by $\frac{p}{q}$, where p and q are relatively prime integers. Find $p + q$.

4. For positive integers n, let $f(n)$ equal the number of subsets of the first 13 positive integers whose members sum to n. Compute

$$\sum_{n=46}^{86} f(n).$$

5. The average of all ten-digit base-ten positive integers $\overline{d_9 d_8 \dots d_1 d_0}$ that satisfy the property $|d_i - i| \le 1$ for all $i \in \{0, 1, \dots, 9\}$ can be written as a common fraction $\frac{p}{q}$, where p and q are relatively prime integers. Compute the remainder when $p + q$ is divided by 10^6.

6. Gary has 2 children. We know one is a boy born on a Friday. Assume birthing boys and girls are equally likely, being born on any day of the week is equally likely, and that these properties are independent of each other, as well as independent from child to child. The probability that both of Gary's children are boys can be expressed as $\frac{a}{b}$ where a and b are relatively prime integers. Find $a + b$.

7. A number line with the integers 1 through 20, from left to right, is drawn. Ten coins are placed along this number line, with one coin at each odd number on the line. A legal move consists of moving one coin from its current position to a position of strictly greater value on the number line that is not already occupied by another coin. How many ways can we perform two legal moves in sequence, starting from the initial position of the coins? Note: different two-move sequences that result in the same position are considered distinct.

8. Each of the 9 cells in a 3×3 grid is colored either blue or white with equal probability. The expected value of the area of the largest square of blue cells contained within the grid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. Let $S = \{1, 2, 3, \dots, 26\}$. Find the number of ways in which S can be partitioned into thirteen subsets such that the following is satisfied:
   - each subset contains two elements of S, and
   - the positive difference between the elements of a subset is 1 or 13.

10. Let P be a set of nine points in the Cartesian coordinate plane, no three of which lie on the same line. Call an ordering $\{Q_1, Q_2, \dots, Q_9\}$ of the points in P *special* if there exists a point C in the same plane such that $CQ_1 < CQ_2 < \cdots < CQ_9$. Over all possible sets P, what is the largest possible number of distinct special orderings of P?

Probability and Combinatorics Solutions

1. Two cases: Plus and minus sign in the second spot, which each occur with a 1/6 chance. In each case, there are 6 valid combinations of numbers for the other three places. Each of those combinations occurs with probability $1/6^3$, so the overall probability is $2 \cdot 6/6^4 = 1/108$, so $p + q = \boxed{109}$.

2. If a positive integer is almost square, then all of its digits must be 0, 1, 4, or 9. Positive integers less than 1000 have 3 digits (in this case, we define two-digit numbers as having a leading 0 and one-digit numbers as having 2 leading 0s), and since there are 4 choices for each of these digits, we narrow our search for almost-squares down to $4^3 = 64$ integers. However, we can further eliminate the integer 0 from our count, since this integer is not positive and therefore not almost square. We have 63 candidates left.

   Now, we employ complementary counting; we find the number of perfect squares less than 1000 whose digits are all perfect squares. A quick check reveals that there are 9 such positive integers $(1, 4, 9, 49, 100, 144, 400, 441,$ and $900)$. Thus, there are $63 - 9 = 54$ almost squares less than 1000, so the number of positive integers less than 1000 that are *not* almost square is $999 - 54 = \boxed{945}$.

3. $\boxed{3}$. In order to visualize the region defined by $|x - \frac{1}{2}| + |y - \frac{1}{2}| = \frac{1}{2}$, let us first set $y = 0$. Thus, since $|y - \frac{1}{2}| = \frac{1}{2}$, x must be $\frac{1}{2}$ in order to force $|x - \frac{1}{2}|$ to be 0. Next, we see that as we linearly increase y from 0 to $\frac{1}{2}$, the term $|y - \frac{1}{2}|$ becomes linearly smaller, and thus x can linearly deviate away from $x = \frac{1}{2}$ in both the positive and negative direction. Then, at $y = \frac{1}{2}$, $|y - \frac{1}{2}| = 0$, and thus $x = 0$ or $x = 1$. If we continue this process for $y = \frac{1}{2}$ to $y = 1$, we find that $|x - \frac{1}{2}| + |y - \frac{1}{2}| = \frac{1}{2}$ defines a square rotated 45°, with diagonal lengths 1, and residing inside the unit square. Thus, since the unit square has area 1, the probability that a point lies within the region we just drew is just the area of the region defined by $|x - \frac{1}{2}| + |y - \frac{1}{2}| < \frac{1}{2}$, or $\frac{1 \times 1}{2} = \frac{1}{2}$, and thus the answer is $1 + 2 = 3$.

4. Let $S = \{1, 2, \ldots, 13\}$. Note that $\sum_{k \in S} k = \sum_{k=1}^{13} k = \frac{13(13+1)}{2} = 91$. Therefore, for any subset $X \subseteq S$, we have $\sum_{k \in X} k + \sum_{k \in S \setminus X} k = \sum_{k \in S} k = 91$, so

$$\sum_{k \in S \setminus X} k = 91 - \sum_{k \in X} k.$$

   Therefore, for every set $X \subseteq S$, exactly one of X or $S \setminus X$ is a set whose elements have a sum between 46 and 91 inclusive. So, exactly half of all the subsets of S have a sum between 46 and 91. Therefore, the desired answer is $\frac{1}{2} \cdot 2^{13}$ minus the number of subsets whose elements have a sum between 87 and 91. This quantity we are subtracting equals the number of subsets whose elements have a sum between 0 and 4. The only such subsets are $\{\}$, $\{1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2\}$, $\{3\}$, and $\{4\}$, making seven in total. Thus, the answer is $\frac{1}{2} \cdot 2^{13} - 7 = \boxed{4089}$.

5. We are allowed to select d_9 from $\{8, 9\}$, d_0 from $\{0, 1\}$, and d_i from $\{i-1, i, i+1\}$ for all $i \in \{1, 2, \ldots, 8\}$. The average we wish to compute is also the expected value of $\sum_{i=0}^{9} d_i \cdot 10^i$, where each digit d_i is selected independently per a uniform distribution over its set of possible values. By linearity of expectation,

$$E\left(\sum_{i=0}^{9} d_i \cdot 10^i\right) = \sum_{i=0}^{9} E(d_i) \cdot 10^i = 8.5 \cdot 10^9 + \sum_{i=1}^{8} i \cdot 10^i + 0.5 \cdot 10^0 = 9376543210.5 = \frac{18753086421}{2}.$$

   Therefore, $p + q = 18753086421 + 2 = 18753086423$, so dividing $p + q$ by 10^6 leaves a remainder of $\boxed{86423}$.

6. $\boxed{40}$. For each child, we care about the child's sex (i.e. whether the child is a boy or a girl), and what day of the week the child is born on. Thus, there are $2 \times 7 = 14$ possible sex/birthday combinations for a child.

   Now, since we know that one of Gary's children is a boy born on Friday, there are initially 14 possible combinations for the other child. However, since the boy born on Friday can either be the first or the second child, there are a total of $14+14-1=27$ possible combinations of children Gary can have such that one is a boy born on Friday. Note that in the expression above, we subtract 1 because we double added the case where both of Gary's children are boys born on Fridays. Of these 27 combinations of children, we now find only the combinations where both are boys. Since, as mentioned above, we subtracted 1 case where both children were boys born on Fridays, we find that 14 of these 27 combinations are such that the other child is a girl, and 13 of these 27 combinations are such that the other child is a boy. Thus, the probability is $\frac{13}{27}$, and the answer is $13+27=40$.

7. There are a total of $1+2+\cdots+10=55$ legal moves from the initial position. If on our first move we move a coin from some position i to the kth smallest unoccupied position larger than i, for some $k \in \mathbb{N}$, then the resulting game position will generate $55-k-m$ legal moves, where m is the number of coins the moving coin "jumped over" to get to its new position. In particular, m must equal $k-1$ by the alternating nature of the coin arrangement in the initial position. From the initial position and for any $k \in \{1,\ldots,10\}$, there are exactly $11-k$ ways to move a coin to the kth smallest unoccupied position larger than the coin's current position, so the total number of sequences of two legal moves is

$$\sum_{k=1}^{10}(11-k)(55-k-(k-1)) = 2\sum_{k=1}^{10}(11-k)(28-k) = 2\left(10\cdot 11\cdot 28 - \frac{(11+28)\cdot 10\cdot 11}{2} + \frac{10\cdot 11\cdot 21}{6}\right)$$

$$= 2\,(3080-2145+385) = 2\cdot 1320 = \boxed{2640}.$$

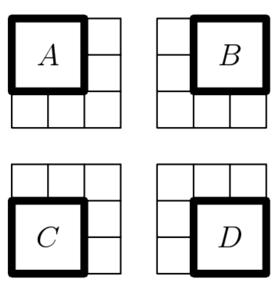
8. Let c be a coloring of the grid, let $A(c)$ be the area of the largest square of blue cells contained within the coloring, and let $P(X)$ denote the probability that event X occurs. We need to compute

$$P\,[A(c)=0]\,(0) + P\,[A(c)=1]\,(1) + [P(A(c)=4]\,(4) + P\,[A(c)=9]\,(9),$$

where $P[A(c)=0]+P[A(c)=1]+P[A(c)=4]+P[A(c)=9]=1$.

There are $2^9=512$ possible colorings of the grid, and clearly, $P[A(c)=0]=\frac{1}{512}$ and $P[A(c)=9]=\frac{1}{512}$.

It now suffices to compute the number of colorings c of the grid such that $A(c)=4$. Note that in order for $A(c)=4$ to be satisfied, at least one of the four 2×2 subgrids contained within the grid must be colored fully blue (these subgrids are shown below and are outlined by bold lines).



Let A be the event that subgrid A is colored fully blue, and define events B, C, and D similarly. Let $|X|$ denote the number of colorings that result when event X occurs. By the Principle of Inclusion-Exclusion, the number of such colorings is

$$\begin{aligned}
&(|A|+|B|+|C|+|D|)\\
&-(|A\cap B|+|A\cap C|+|A\cap D|+|B\cap C|+|B\cap D|+|C\cap D|)\\
&+(|A\cap B\cap C|+|A\cap B\cap D|+|A\cap C\cap D|+|B\cap C\cap D|)\\
&-2(|A\cap B\cap C\cap D|).
\end{aligned}$$

We subtract $2(|A\cup B\cup C\cup D|)$ to exclude the case in which A, B, C, and D are colored fully blue, as the whole grid would be colored blue as a result, meaning that $A(c)=9$; this case is counted 4 times, subtracted 6 times, and counted back in 4 times, so subtracting it 2 times would ensure that it will be excluded from the count. It is routine to compute the number of colorings for each of these cases, and we obtain

$$(2^5 + 2^5 + 2^5 + 2^5) - (2^3 + 2^3 + 2^2 + 2^2 + 2^3 + 2^3) + (2^1 + 2^1 + 2^1 + 2^1) - 2(1) = 94$$

as the overall number of colorings.

Therefore, $P[A(c) = 0] = \frac{1}{512}$, $P[A(c) = 4] = \frac{94}{512}$, and $P[A(c) = 9] = \frac{1}{512}$, so it follows that $P[A(c) = 1] = 1 - \frac{96}{512} = \frac{416}{512}$; then, the desired expected value is

$$\frac{1}{512}(0) + \frac{416}{512}(1) + \frac{94}{512}(4) + \frac{1}{512}(9) = \frac{801}{512}.$$

Hence, $m + n = \boxed{1313}$

9. We arrange the elements of S into a 2×13 grid, as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13
14	15	16	17	18	19	20	21	22	23	24	25	26

Then, a 1×2 tile of this grid will consist of two numbers whose positive difference is either 1 or 13 Thus, any tiling of the 2×13 grid with 1×2 tiles corresponds to a unique desirable partition of S. The only desirable partition of S that cannot be represented by such a tiling occurs when one member of the partition is $\{13, 14\}$. The remainder of such a partition that includes $\{13, 14\}$ must correspond to a tiling of the 2×13 grid with squares 13 and 14 removed. One can quickly deduce that there is only one valid way to tile this figure; that the only desirable partition with $\{13, 14\}$ as a member is $\{\{1, 2\}, \{3, 4\}, \ldots, \{11, 12\}, \{13, 14\}, \{15, 16\}, \ldots, \{25, 26\}\}$.

Therefore, the problem boils down to finding the number of ways to tile a 2×13 grid with 1×2 tiles and then adding 1 to this count. More generally, the number of ways to tile a $2 \times n$ grid with 1×2 tiles is F_{n+1}, where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ (i.e., the $(n + 1)$-th Fibonacci number); this can be shown via a recursive argument. Therefore, the answer is $F_{14} + 1 = \boxed{378}$.

10. Arbitrarily label the points in P as P_1, P_2, …, and P_9. For distinct indices $i, j \in \{1, 2, \ldots, 9\}$, let p_{ij} denote the perpendicular bisector of $\overline{P_iP_j}$. For any two points P_i and P_j, a third point C (in the same plane) is closer to P_i than P_j if and only if C and P_i lie on the same side of p_{ij}. Similarly, C is closer to P_j than P_i if and only if C and P_j lie on the same side of p_{ij}. If we let $L = \{p_{ij} : 1 \leq i < j \leq 9\}$ be the set of all $\binom{9}{2} = 36$ perpendicular bisectors, then the lines in L partition the plane into disjoint 2-D regions. Each region corresponds to a unique special ordering $\{Q_1, Q_2, \ldots, Q_9\}$, such that a point C satisfies $CQ_1 < CQ_2 < \cdots < CQ_9$ if and only if C is strictly inside the region. Thus, the number of distinct special orderings of P equals the number of regions in the partition.

Each triplet of points in P does not lie on a single line (by the assumptions in the problem statement), so each triplet has a single circumcircle, and the three perpendicular bisectors generated by this triplet will concur at the circle's center. This concurrency will limit the number of 2-D regions created by L. Normally, we can slice the plane into 7 regions if we have three lines that intersect in pairs at three different points. However, if three lines concur at a single point, then the number of regions is reduced to 6. In general, each concurrency point of three lines, among a finite collection of many lines in the plane, removes one 2-D region from the count. This holds because we can "perturb" any one of the three lines, shifting it by an arbitrarily small amount, so that the three lines no longer concur and a new triangular region opens up between them. This observation motivates the following approach: count the maximum possible number of regions that can be created from $\binom{9}{2} = 36$ lines, and then subtract the number of points at which three of those lines concur. This optimal number of regions is feasible because it is definitely possible to ensure that (1) no two lines in L are parallel and (2) no more than three lines in L concur at any point; in fact, if the points in P are chosen according to some continuous random distribution, then (1) and (2) will generally hold with probability 1.

So, we can achieve the minimum $\binom{9}{3} = 84$ concurrency points, since we can have a distinct circumcenter for each triplet of points in P and no other lines in L unintentionally passing through the circumcenter. Also, the maximum number of regions created by n lines is given by $\frac{n(n+1)+2}{2}$. There are $\binom{9}{2} = 36$ lines in L, so our final count is

$$\frac{36(36+1)+2}{2} - \binom{9}{3} = 18 \cdot 37 + 1 - 84 = 666 - 83 = \boxed{583}.$$

Remark: From our analysis, a general formula for the maximum number of special orderings of a set of n points in the plane is $\frac{1}{2}\left(\binom{n}{2}\left(\binom{n}{2}+1\right)+2\right) - \binom{n}{3}$. These values are documented in sequence A308305 in OEIS.